International Journal of Research in Engineering and Science (IJRES)
ISSN (Online): 2320-9364, ISSN (Print): 2320-9356
www.ijres.org Volume 7 Issue 6 ǁ 2019 ǁ PP. 06-20

New Escape Criteria for Complex Fractals Generation in Jungck-Cr Orbit

Arjun Singh. Muktanjali Mishra, Sunita Priyadarsini, Bibekananda Biswal
Department of Mathematics, NM Institute of Engineering and Technology,Bhubaneswar , Odisha
Department of Mathematics, Raajdhani Engineering College,Bhubaneswar,Odisha
Department of Mathematics,Aryan Institute of Engineering and Technology Bhubnaeswar , Odisha
Department of Basic Science & Humanities ,Capital Engineering College,Bhubaneswar,Odisha

ABSTRACT; *In recent years, researchers have studied the use of different iteration processes from fixed point theory for the generation of complex fractals. Examples are the Mann, the Ishikawa, the Noor, the Jungck-Mann and the Jungck-Ishikawa iterations. In this paper, we present a generalization of complex fractals, namely Mandelbrot, Julia and multicorn sets, using the Jungck-CR implicit iteration scheme. This type of iteration does not reduce to any of the other iterations previously used in the study of complex fractals; thus, this generalisation gives rise to new fractal forms.* $z^n - az + c$, where a, c C, *We prove a new escape criterion for a polynomial of the following form and present some graphical examples of the obtained complex fractals.*
Key words : Julia set; Mandelbrot set; Jungck-CR orbit; escape criteria.

I. INTRODUCTION

A very common method of generating fractal patterns in the complex plane is the repeated iteration of a complex function f : C C in the following way:

$$z_{n+1} = f(z_n), \tag{1}$$

where f depends on some constant c C and z_0 C is the point from a considered area of the complex plane [38]. The two most famous examples of such fractals are the Mandelbrot and Julia sets.

The Mandelbrot and Julia sets are some of the best known illustrations of a highly complicated chaotic system generated by a very simple mathematical process. They were introduced by Benoit Mandelbrot in the late 1970s [27], but Julia sets were studied much earlier, namely in the early twen- tieth century by French mathematicians Pierre Fatou [10] and Gaston Julia [17]. While working at IBM, Mandelbrot studied their works and plotted the Julia sets for . $z^2 +$ .Building on them, he he plotted the Mandelbrot set. He was surprised by his results. Since then, many mathematicians have studied different properties of the Mandelbrot and Julia sets and proposed accordingly various gen- eralisations. The first and most obvious generalisation was the use of the $z^p + c$ function instead of the quadratic one used by Mandelbrot [8, 26]. Further, additional types of functions were studied: rational [32], transcendental [9], elliptic [24], anti-polynomials [6] etc. Another development, build- ing on the study of the Mandelbrot and Julia sets, was the extension from complex numbers to other algebras, e.g. quaternions [7], octonions [15], bicomplex numbers [23] etc.

In recent years, some researchers have focused on a different kind of generalisation. They used results that can be found in fixed point theory. In this theory, we can find methods of locating fixed points that replace the feedback process (1) with other types of iteration processes. The use of differ- ent iteration processes began in 2004 in the works of Rani and Kumar [36, 37]. They used the Mann iteration and proved new escape criteria for the generation of the Mandelbrot and Julia sets using this type of iteration. Further studies on the use of different iteration processes were conducted, focusing on two in particular. The first type, like the Mann iteration used by Rani and Kumar, was an explicit process. Examples of processes of this type that were used to generate the

Mandelbrot and Julia sets are the following: Ishikawa [5, 14], Noor [2], S [22] and Abbas [25]. The second type of iteration process is an implicit one. In the literature, we can find the use of the Jungck-Mann [28], Jungck-Ishikawa [28] and Jungck-Noor [21] iterations, which are examples of such processes. Moreover, studies on the noise-perturbed versions of the Mandelbrot and Julia sets generated by the different iteration processes were conducted [1, 29, 35].

The Mandelbrot and Julia sets are not only interesting from a mathematical point of view. They have applications in other fields also, e.g. physics [3], biology [4] and robotics [45]. One of the most natural applications of the Mandelbrot and Julia sets – because of their beauty – was their use in computer graphics. The sets were used as a source of aesthetic patterns [44, 46], for creating realistic phenomena and landscapes [11] or for image manipulation [40]. In [42], Sun et al . have used Julia sets

NEW ESCAPE CRITERIA FOR COMPLEX FRACTALS 1287

as the base for creating a dictionary with domain blocks used in fractal image compression. Moreover, in [41], Sun et al. have proposed an image encryption algorithm in which Julia set parameters are used to generate a random sequence as the initial keys for the algorithm. Another example of complex fractals generated using (1) are fractals obtained with the help of complex polynomial root finding methods, known as polynomiography [19]. In this type of fractals we visualise – in different ways – the process of root finding of a given polynomial. The literature describes various root finding methods, which contribute to the great number of fractal patterns ob-tained through them. Moreover, in the case of polynomiography a study on the use of various iteration processes was conducted [12, 13, 20, 34], which allowed to extend the variety of the obtained fractal patterns.

In this paper, we focus only on the Mandelbrot and Julia sets. We define the Jungck-CR orbitand show how to use it to generalise the feedback process used to generate Mandelbrot and Julia sets. Moreover, we prove the escape criterion for a complex polynomial of the form

$z^m - az + c,$

, where a,c C.

The paper is organised as follows. In Section 2, we give definitions for the Julia, Mandelbrot and multicorn sets together with the escape time algorithm for their generation. Moreover, we define some of the iterations that were previously used in the study of complex fractals. In Section 2, we define the Jungck-CR iteration and we discuss how to use this type of iteration for the generation of complex fractals. Moreover, we derive the escape criteria for the m-th degree polynomial. In Section 3, we present some examples of Julia, Mandelbrot and multicorn sets in Jungck-CR orbit obtained with the escape time algorithm. Finally, in Section 5, we present our concluding remarks.

II. PRELI MINARIES

In this section, we provide the preliminary definitions that form the basis of our work.

filled Julia set of the polynomial function $Q_c : C \quad C$ is connected, i.e.

$$M = \{c \quad C : K_{Q_c} \text{ is connected } \}. \tag{3}$$

The Mandelbrot set can be equally defined in the following way [43].

$$M = \{c \quad C : |Q_c^n(z^*)| \quad 8 \quad \text{as } n \quad 8\}, \tag{4}$$

where z^* is any critical point of Q_c, i.e. $Q_c(z^*) = 0$.

Definition 3 — (Multicorn). Let $A_c(z) = \quad z^m + c$, where $c \quad C$. The multicorn M^* for A_c is defined as the collection of all $c \quad C$ to which the orbit of 0 under the action of A_c is bounded, i.e.

$$M^* = \{c \quad C : |A_c^n(0)| \quad 8 \quad \text{as } n \quad 8\}. \tag{5}$$

Definition 1 — (Julia Set). Let Q_c : C C be a polynomial function that depends on c C. The filled Julia set K_{Q_c} of the function Q_c is defined as

$$K_{Q_c} = \{z \quad C : |Q_c^n(z)| \quad 8 \quad \text{as } n \quad 8\}, \tag{2}$$

where $Q_c^n(z)$ is the n-th iterate of the function Q_c. The Julia set J_{Q_c} of the function Q_c is defined as the boundary of K_{Q_c}, i.e. $J_{Q_c} = K_{Q_c}$.

Definition 2 — (Mandelbrot set). The Mandelbrot set M consists of all parameters c to which the

The multicom for $m = 2$ is called the tricom.

The most widely used algorithm to generate images of Julia, Mandelbrot and multicorn sets is the escape time algorithm. The colour of each point in the algorithm is determined based on the number of iterations necessary to evaluate whether the orbit sequence tends to infinity or not. In order to establish whether the orbit escapes or not we use the escape criterion. For instance, for the classical Mandelbrot and Julia sets, i.e. sets defined by

Mandelbrot and Julia sets, i.e. sets defined by $Q_c(z) = z^2 + c$, the escape criterion is the following: if there exists $k = 0$ such that

$$|Q_c^k(z)| > \max\{|c|, 2\}, \tag{6}$$

then $Q_c^n(z) \; 8$ as $n \; 8$.

We call the right side of (6) the escape threshold (or bailout value). This threshold can be different for different functions Q_c and plays a very important role in the generation of Mandelbrot and Julia sets.

The escape time algorithms for the generation of Mandelbrot and Julia sets are presented in Algorithm 1 and 2 respectively. For the generation of multicoms, we can use the same algorithm as for the Mandelbrot set, but with A_c instead of Q_c and the critical point set to 0.

In fixed point theory, there are many theorems and methods that allow one to find fixed points in a given mapping. One of the main foci of the theory is the iterative approximation of fixed points. For this, we use different kinds of iteration processes. Let us recall some of them.

NEW ESCAPE CRITERIA FOR COMPLEX FRACTALS 1289

Algorithm 1 Mandelbrot set generation

Require: Q_c : C C – polynomial function, A C – area, K – the maximum number of iterations, colourmap[0..C - 1] – colourmap with C colours.
Ensure: Mandelbrot set for the area A.

```
for c  A do
    R = calculate escape threshold
    n = 0
    z_0 = critical point of Q_c
    while n = K do
        z_{n+1} = Q_c (z_n )
        if |z_{n+1} | > R then
            break
        end if
        n = n + 1
    end while
    i = (C - 1) n/K
    colour c with colourmap[i]
end for
```

Algorithm 2 Julia set generation

Require: Q_c : C C – polynomial function, c C – parameter, A C – area, K – the maximum number of iterations, colourmap[0..C - 1] – colourmap with C colours.
Ensure: Julia set for the area A.

```
R = calculate escape threshold
for z_0  A do
    n = 0
    while n = K do
        z_{n+1} = Q_c (z_n )
        if |z_{n+1} | > R then
            break
        end if
        n = n + 1
    end while
    i = (C - 1) n/K
    colour z_0 with col ourmap[i]
end for
```

Definition 4 — (Picard iteration [33]). Let $T : X \quad X$ be a mapping on a metric space (X, d), where d is a metric and let $x_0 \quad X$ be a starting point. The Picard iteration is defined as follows:

$$x_{n+1} = T(x_n), \; n = 0, 1, 2, \ldots \tag{7}$$

The sequence $\{x_n\}_{n \quad N}$ is called the Picard orbit of x_0.

Definition 5 — (Jungck iteration [18]). Let $S, T : X \quad X$ be mappings on a metric space (X, d), where d is a metric and let $x_0 \quad X$ be a starting point. The Jungck iteration is defined as follows:

$$S(x_{n+1}) = T(x_n), n = 0, 1, 2, \dots \tag{8}$$

The sequence $\{x_n\}_n \quad \mathrm{N}$ is called the Jungck orbit of x_0.

Definition 6 — (Jungck-Mann iteration [39]). Let $S, T : X \quad X$ be mappings on a metric space (X, d), where d is a metric and let $x_0 \quad X$ be a starting point. The Jungck-Mann iteration is defined as follows:

$$S(x_{n+1}) = (1 - \alpha)S(x_n) + \alpha T(x_n), n = 0, 1, 2, \dots, \tag{9}$$

where $\alpha \quad (0, 1]$. The sequence $\{x_n\}_n \quad \mathrm{N}$ is called the Jungck-Mann orbit of x_0.

Let us notice that the Jungck-Mann iteration reduces to the Jungck iteration if $\alpha = 1$, and to the Picard iteration if $S(x) = x$ and $\alpha = 1$.

Definition 7 — (Jungck-Ishikawa iteration [31]). Let $S, T : X \quad X$ be mappings on a metric space (X, d), where d is a metric and let $x_0 \quad X$ be a starting point. The Jungck-Ishikawa iteration is defined as follows:

$$\begin{aligned} S(x_{n+1}) &= (1 - \alpha)S(x_n) + \alpha T(y_n), \\ S(y_n) &= (1 - \beta)S(x_n) + \beta T(x_n), n = 0, 1, 2, \dots, \end{aligned} \tag{10}$$

where $\alpha \quad (0, 1]$ and $\beta \quad [0, 1]$. The sequence $\{x_n\}_n \quad \mathrm{N}$ is called the Jungck-Ishikawa orbit of x_0.

Let us notice that the Jungck-Ishikawa iteration reduces to the Jungck-Mann iteration if $\beta = 0$.

Definition 8 — (Jungck-Noor iteration [30]). Let $S, T : X \quad X$ be mappings on a metric space (X, d), where d is a metric and let $x_0 \quad X$ be a starting point. The Jungck-Noor iteration is defined as follows:

$$\begin{aligned} S(x_{n+1}) &= (1 - \alpha)S(x_n) + \alpha T(y_n), \\ S(y_n) &= (1 - \beta)S(x_n) + \beta T(u_n), \\ S(u_n) &= (1 - \quad)S(x_n) + \quad T(x_n), n = 0, 1, 2, \dots, \end{aligned} \tag{11}$$

where $\alpha \quad (0, 1]$ and $\beta, \quad [0, 1]$. The sequence $\{x_n\}_n \quad \mathrm{N}$ is called the Jungck-Noor orbit of x_0.

Let us notice that the Jungck-Noor iteration reduces to the Jungck-Ishikawa iteration if $= 0$. This is the most general iteration of the above.

Let us notice that the Jungck-Noor iteration reduces to the Jungck-Ishikawa iteration if = 0.
This is the most general iteration of the above.

III. NEW ESCAPE CRITERIA FOR COMPLEX FRACTALS 1291

3. MANDELBROT AND JULIA SETS IN JUNGCK-CR ORBIT

Let us notice that the Picard iteration is the iteration used in the generation of complex fractals. In the literature, we can find studies that have replaced the Picard iteration with other iterations, e.g. with the ones presented in Section 2 [21, 28]. In this section, we show how to use one of the implicit iterations, namely the Jungck-CR iteration, for the generation of Julia, Mandelbrot and multicorn sets.
Let us start with the definition of a Jungck-CR iteration and its orbit.

Let us start with the definition of a Jungck-CR iteration and its orbit.

Definition 9 — (Jungck-CR iteration [16]). Let $S, T : X \quad X$ be mappings on a metric space (X, d), where d is a metric and let $x_0 \quad X$ be a starting point. The Jungck-CR iteration is defined as follows:

$$
\begin{aligned}
S(x_{n+1}) &= (1 - \alpha)S(y_n) + \alpha T(y_n),\\
S(y_n) &= (1 - \beta)T(x_n) + \beta T(u_n),\\
S(u_n) &= (1 - \quad)S(x_n) + \quad T(x_n),\ n = 0, 1, 2, \ldots,
\end{aligned}
\tag{12}
$$

where $\alpha \quad (0, 1]$ and $\beta, \quad [0, 1]$. The sequence $\{x_n\}_{n \quad \mathrm{N}}$ is called the Jungck-CR orbit of x_0.

Let us notice that the Jungck-CR iteration does not reduce to any of the discussed iterations: Picard, Jungck-Mann, Jungck-Ishikawa, Jungck-Noor. Thus, using this iteration creates a completely new orbit and, by consequence, new fractal sets. In the Picard orbit we use only one mapping and in Jungck-CR we have two mappings. Thus, if we want to replace the Picard orbit with the Jungck-CR orbit, we need to take into account the different number of mappings in the iterations. We handle this in the following way. Let

In the Picard orbit we use only one mapping and in Jungck-CR we have two mappings. Thus, if we want to replace the Picard orbit with the Jungck-CR orbit, we need to take into account the different number of mappings in the iterations. We handle this in the following way. Let $Q_c : \mathrm{C} \quad \mathrm{C}$ be a polynomial function. We deconstruct Q_c into two mappings S, T in such a way that $Q_c = T - S$ and S is injective. In the case of multicorns, we deconstruct $Q_c^*(z) = \quad Q_c(z)$ in the following way: $Q_c^* = T - S$, where $T = A_c$ and S is injective. Of course, this type of deconstruction restricts the choice of the polynomial functions that can be used. With the deconstruction, we also need to derive a new escape criterion for the mappings and (12).

In the following subsection we prove the escape criterion for a class of polynomials.

3.1 *Escape criterion for a complex polynomial of the form* $z^m - az + c$

Let $Q_c(z) = z^m - az + c$, where $m \quad \{2, 3, \ldots\}$ and $a, c \quad \mathrm{C}$. We deconstruct Q_c in the following way: $T(z) = z^m + c$ and $S(z) = az$.

***Theorem* 1** — *Assume that* $|z| = |c| > \left(\frac{2(1+|a|)}{\alpha}\right)^{\frac{1}{m-1}}$, $|z| = |c| > \left(\frac{2(1+|a|)}{\beta}\right)^{\frac{1}{m-1}}$, $|z| = |c| >$

$2(1+ \ |a|) \quad ^{\frac{1}{m-1}}$, *where* $a, \beta,$ $(0, 1]$ *and define* $\{z_n\}_{n \ N}$ *as follows:*

$$
\begin{aligned}
S(z_{n+1}) &= (1 - a)S(y_n) + aT(y_n), \\
S(y_n) &= (1 - \beta)T(z_n) + \beta T(u_n), \\
S(u_n) &= (1 - \ )S(z_n) + \ T(z_n), n = 0, 1, 2, \ldots,
\end{aligned}
\tag{13}
$$

where $z_0 = z$. *Then*, $|z_n| \quad 8$ as $n \quad 8$.

PROOF : Because $T(z) = z^m + c$, $S(z) = az$ and $z_0 = z$, we have

$$
\begin{aligned}
|S(u_0)| &= |(1 - \ )S(z) + \ T(z)| = |(1 - \ )az + \ (z^m + c)| \\
&= |z^m + c| - (1 - \ )|az| = |z^m| - |c| - |az| + |az| \\
&= |z^m| - |z| - |a||z| \qquad (\text{because } |z| = |c| \text{ and } = 1).
\end{aligned}
$$

The above expression gives us

$$
|au_0| = |z^m| - (1 + |a|)|z| = |z|(|z|^{m-1} - (1 + |a|)).
$$

Thus

$$
|u_0| = |z| \ |z|^{m-1} \ 1 + |a| - \ 1 \ .
$$

In the second step of the iteration we have

$$
\begin{aligned}
|S(y_0)| &= |(1 - \beta)T(z) + \beta T(u_0)| = |(1 - \beta)(z^m + c) + \beta(u_0^m + c)| \\
&= |(1 - \beta)z^m + \beta u_0^m| - |c| = \beta|u_0^m| - \beta|z^m| + |z^m| - |c|.
\end{aligned}
$$

Since $|z| > \ 2(1+ \ |a|) \ ^{\frac{1}{m-1}}$, which implies $|z|^m \ |z| \ \frac{m-1}{1+|a|} - 1 \ ^m > |z|^m$, hence $|u_0|^m > |z|^m$ $\frac{|z|^2}{1+|a|} - 1 \ ^m = |z|^m = |z|^m$. We get

$$
\begin{aligned}
|ay_0| &= (\beta - \beta + 1) \ |z^m| - |c| \\
&= \beta \ |z^m| - |z| \qquad (\text{because } |z| = |c| \text{ and } 1 - \beta = 0) \\
&= \beta \ |z^m| - (1 + |a|)|z| \qquad (\text{because } 1 + |a| = 1).
\end{aligned}
$$

Thus

$$
|y_0| = |z| \beta |z| \ ^{m-1} \ 1 + |a| - \ 1 \ .
$$

Since $|z| = |c| > \left(\frac{2(1+|a|)}{\beta}\right)^{\frac{1}{m-1}}$ and $|z| = |c| > \left(2(1+|a|)\right)^{\frac{1}{m-1}}$, so $|z| = \left(\frac{2(1+|a|)}{\beta}\right)^{\frac{1}{m-1}}$, thus $|z|^{m-1} = |z|^{m-1}\left(\frac{\beta|z|^{m-1}}{1+|a|} - 1\right)^{m-1}$. Hence

$$|y_0|^{m-1} = |z|^{m-1}\left(\frac{\beta|z|^{m-1}}{1+|a|} - 1\right)^{m-1} = |z|^{m-1} = \beta|z|^{m-1}.$$

In the third step of the iteration we have

$$\begin{aligned}
|S(z_1)| &= |(1-\alpha)S(y_0) + \alpha T(y_0)| = |(1-\alpha)ay_0 + \alpha(y_0^m + c)| \\
&= |(1-\alpha)ay_0 + \alpha y_0^m + \alpha c| = \alpha|y_0^m| - (1-\alpha)|ay_0| - \alpha|c| \\
&= \alpha|y_0^m| - |a||y_0| + \alpha|a||y_0| - \alpha|c| \\
&= \alpha|y_0^m| - |a||y_0| - |y_0| \quad (\text{because } |y_0| = |z| = |c|) \\
&= |y_0|(\alpha|y_0|^{m-1} - (1+|a|)).
\end{aligned}$$

Moreover, $|S(z_1)| = \alpha z_1$. Thus

$$|z_1| = |y_0|\left(\frac{\alpha|y_0|^{m-1}}{1+|a|} - 1\right).$$

Because $|y_0| = |z|$ and $|z| = \left(\frac{2(1+|a|)}{\alpha}\right)^{\frac{1}{m-1}}$, thus $\frac{\alpha|y_0|^{m-1}}{1+|a|} = \frac{\alpha|z|^{m-1}}{1+|a|} > 2$. Therefore, $|y_0|\left(\frac{\alpha|y_0|^{m-1}}{1+|a|} - 1\right) = |z|\left(\frac{\alpha\beta|z|^{m-1}}{1+|a|} - 1\right)$. By consequence

$$|z_1| = |z|\left(\frac{\alpha\beta|z|^{m-1}}{1+|a|} - 1\right).$$

Since $|z| > \left(\frac{2(1+|a|)}{\alpha}\right)^{\frac{1}{m-1}}$, $|z| > \left(\frac{2(1+|a|)}{\beta}\right)^{\frac{1}{m-1}}$ and $|z| > \left(2(1+|a|)\right)^{\frac{1}{m-1}}$, so $|z| > \left(\frac{2(1+|a|)}{\alpha\beta}\right)^{\frac{1}{m-1}}$. By consequence, $\frac{\alpha\beta|z|^{m-1}}{1+|a|} - 1 > 1$. Therefore, there exists > 0 such that $\frac{\alpha\beta|z|^{m-1}}{1+|a|} - 1 > 1 +$. Consequently, $|z_1| > (1+\ )|z|$. In particular, $|z_1| > |z|$. Thus, we may apply the same argument repeatedly to find $|z_n| > (1+\ )^n|z|$. This way, the orbit of z tends to infinity, which completes the proof. 2

Corollary 1 — Suppose that

$$|c| > \left(\frac{2(1+|a|)}{\alpha}\right)^{\frac{1}{m-1}}, |c| > \left(\frac{2(1+|a|)}{\beta}\right)^{\frac{1}{m-1}} \text{ and } |c| > \left(2(1+|a|)\right)^{\frac{1}{m-1}}, \qquad (14)$$

then the Jungck-CR orbit escapes to infinity.

Corollary 2 — (Escape Criterion). Let $\alpha, \beta, \gamma \in (0,1]$ and

$$|z| > \max\left\{|c|, \left(\frac{2(1+|a|)}{\alpha}\right)^{\frac{1}{m-1}}, \left(\frac{2(1+|a|)}{\beta}\right)^{\frac{1}{m-1}}, \left(\frac{2(1+|a|)}{\gamma}\right)^{\frac{1}{m-1}}\right\}, \tag{15}$$

then there exists $\lambda > 0$, such that $|z_n| > (1+\lambda)^n |z|$ and $|z_n| \to \infty$ as $n \to \infty$.

Corollary 3 — Suppose that

$$|z_k| > \max\left\{|c|, \left(\frac{2(1+|a|)}{\alpha}\right)^{\frac{1}{m-1}}, \left(\frac{2(1+|a|)}{\beta}\right)^{\frac{1}{m-1}}, \left(\frac{2(1+|a|)}{\gamma}\right)^{\frac{1}{m-1}}\right\}, \tag{16}$$

for some $k \geq 0$. Then, there exists $\lambda > 0$, such that $|z_{k+n}| > (1+\lambda)^n |z_k|$ and $|z_n| \to \infty$ as $n \to \infty$.

IV. EXAMPLE

In this section, we present some examples of the Julia, Mandelbrot and multicorn sets images ob-tained using the Jungck-CR orbit and the escape criterion derived in Section 3. The images were generated using the escape time algorithm that was implemented in Mathematica. The times required for generating the images were between 5 and 7 seconds on a computer with an Intel Core i5 (2 GHz) processor and 3 GB of RAM.

Examples of Julia sets in Jungck-CR orbit

The first example presents Julia sets for $Q_c(z) = z^3 + (3/2)z + c$ and $c = 1.75i$ generated using Jungck and Jungck-CR iterations. The obtained images are presented in Figure 1 and the parameters used to generate them were the following:

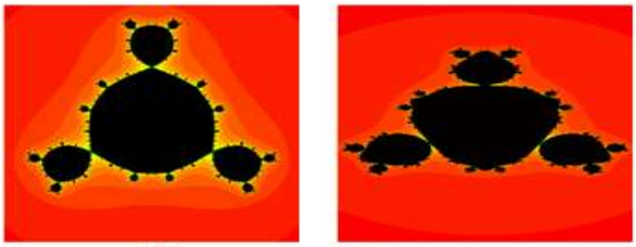
(a) $A = [-1.7, 1.7]^2$, $K = 50$, Jungck iteration.

(b) $A = [-1.5, 1.5] \times [-2.5, 2.5]$, $K = 50$, Jungck-CR iteration with $\alpha = \beta = \gamma = 0.9$.

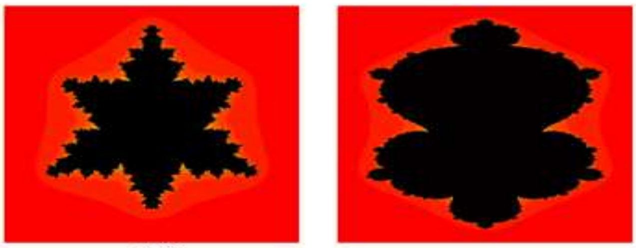
(c) $A = [-2.5, 2.5]^2$, $K = 50$, Jungck-CR iteration with $\alpha = 0.9$, $\beta = 0.5$, $\gamma = 0.1$.

(d) $A = [-3.5, 3.5] \times [-6, 6]$, $K = 50$, Jungck-CR iteration with $\alpha = \beta = \gamma = 0.1$.

NEW ESCAPE CRITERIA FOR COMPLEX FRACTALS 1295



(a) (b)



(c) (d)

Figure 1: Julia set for $Q_c(z) = z^3 + (3/2)z + c$ and $c = 1.75i$ in (a) Jungck and (b)-(d) Jungck-CR orbit

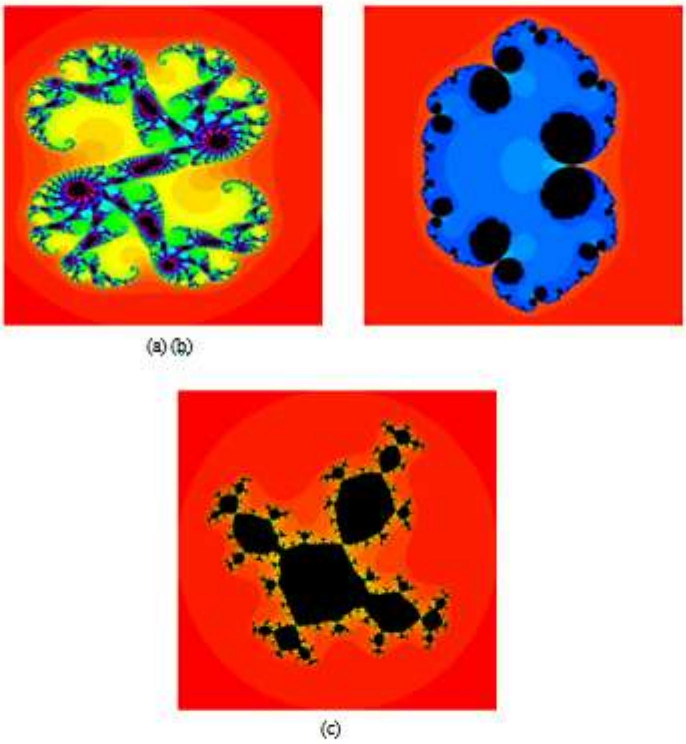


Figure 2: Julia sets for various parameters in Jungck-CR orbit

From the obtained images we can observe that the Julia sets generated using the Jungck-CR iteration differ in a significant way from the one obtained with the Jungck iteration. The shape of the set in Figure (b) is somewhat similar to the one from Figure (a), but the shapes of the sets from Figure (c) and (d) are completely different. The change of the parameters in the Jungck-CR iteration allows us to obtain very different shapes of Julia sets for the same polynomial used to generate them. Moreover, when we look at the area that the sets occupy, we see that the sets obtained with the Jungck-CR iteration occupy a larger area than the set generated using the Jungck iteration. In the second example we present various Julia sets generated using the Jungck-CR iteration. In order to generate the images of Julia sets from Figure 2 the following parameters were used:

(a) $Q_c(z) = z^2 + 2z + c$, $c = 17.28 + 5.5i$, $A = [-28, 14] \times [-28, 28]$, $K = 50$, Jungck-CR iteration with $\alpha = \beta = \ = 0.1$.

(b) $Q_c(z) = z^3 + (3/2)z + c$, $c = 0.72$, $A = [-2.3, 2.3]^2$, $K = 50$, Jungck-CR iteration with $\alpha = \beta = \ = 0.5$.

(c) $Q_c(z) = z^4 + 3z + c$, $c = 3.9 + 7i$, $A = [-2.5, 2.5]^2$, $K = 50$, Jungck-CR iteration with $\alpha = \beta = \ = 0.5$.

From the images, we see that using different combinations of the parameters, e.g. polynomial, parameters in the Jungck-CR iteration etc., we are able to obtain very diverse fractal patterns that have potential artistic applications.

4.2 Examples of Mandelbrot sets in the Jungck-CR orbit We begin with an example presenting the use of Jungck and Jungck-CR iterations. In the example, we use $Q_c(z) = z^4 + 3z + c$. The parameters used to generate the images from Figure 3 were the following:

(a) $A = [-7, 6] \times [-6, 6]$, $K = 20$, Jungck iteration.

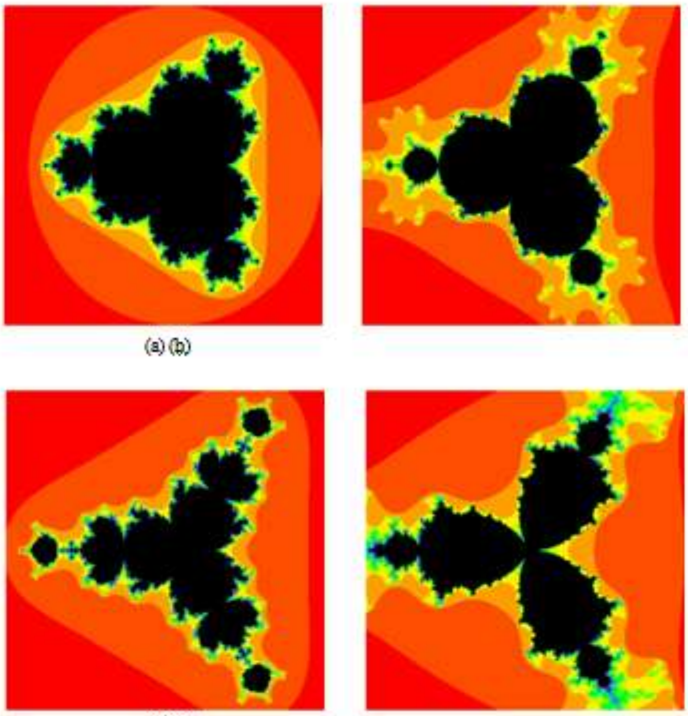
(b) $A = [-8, 8]^2$, $K = 20$, Jungck-CR iteration with $\alpha = 0.5$, $\beta = 0.9$, $= 0.9$.

(c) $A = [-13, 10] \times [-11, 11]$, $K = 20$, Jungck-CR iteration with $\alpha = 0.9$, $\beta = 0.9$, $= 0.1$.

(d) $A = [-63, 60] \times [-61, 61]$, $K = 20$, Jungck-CR iteration with $\alpha = \beta = \ = 0.1$.

1.Looking at the images in Figure 3 we can observe that the shape of the Mandelbrot set is different
in the Jungck-CR orbit and the Jungck orbit. The number of bulbs in all the images is the same – three – but their shape changes with the change of the parameters in the Jungck-CR iteration. We can also observe that for different values of the a , ß and parameters the three-fold symmetry of the sets remains unchanged. Moreover, the area that the set occupies changes for different values of the parameters used in the Jungck-CR iteration.

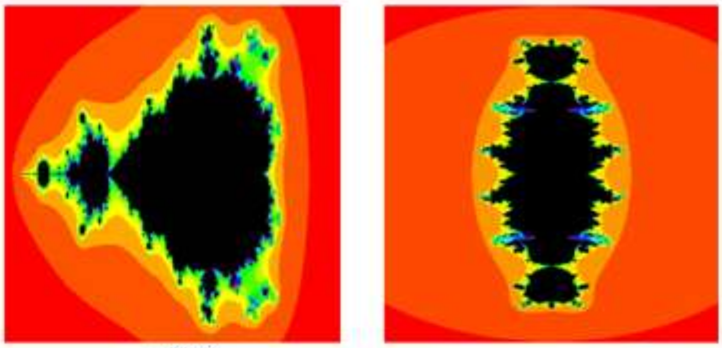
NEW ESCAPE CRITERIA FOR COMPLEX FRACTALS 1297



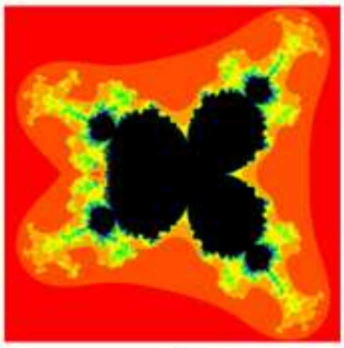
(a) (b)

(c) (d)

Figure 3: Mandelbrot set for $Q_c(z) = z^4 + 3z + c$ in (a) Jungck and (b)-(d) Jungck-CR orbit



(a) (b)



(c)

Figure 4: Mandelbrot sets for various parameters in the Jungck-CR orbit

Figure 4 presents examples of various Mandelbrot sets generated using the Jungck-CR iteration and the following parameters:

(a) $Q_c(z) = z^2 + 2z + c$, $A = [-35.5, 12] \times [-12, 12]$, $K = 20$, Jungck-CR iteration with $\alpha = 0.6$, $\beta = 0.7$, $= 0.2$.

(b) $Q_c(z) = z^3 + (3/2)z + c$, $A = [-3.5, 3.5] \times [-5.5, 5.5]$, $K = 20$, Jungck-CR iteration with $\alpha = 0.1$, $\beta = 0.5$, $= 0.7$.

(c) $Q_c(z) = z^{4.5} + 3z + c$, $A = [-12, 12]^2$, $K = 20$, Jungck-CR iteration with $\alpha = \beta = \ = 0.5$.

Similar to the case of the Julia sets, we can observe from the presented images that using the Jungck-CR iteration we are able to obtain various shapes of the corresponding Mandelbrot sets. More- over, the change of the a, ß and parameters in the Jungck-CR iteration has great impact on the shape of the resulting set.

4.3Examples of multicorns in the Jungck-CR orbit

Similar to the case of the Julia and Mandelbrot sets, we begin with an example of multicorns generated with the Jungck and Jungck-CR iterations. In the example, we present multicorns for $Q_c^*(z) = \bar{z}^4 + 3\bar{z} + c$. The obtained images of multicorns are presented in Figure 5. For their generation, the following parameters were used:

(a) $A = [-7, 6] \times [-6, 6]$, $K = 20$, Jungck iteration.

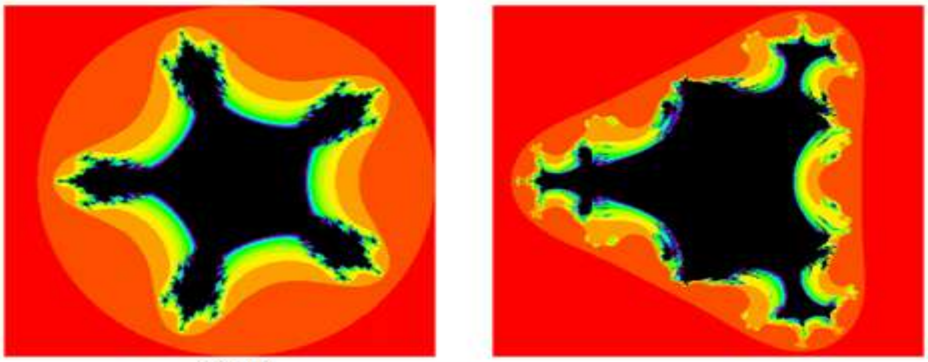
(b) $A = [-8, 8]^2$, $K = 20$, Jungck-CR iteration with $\alpha = \beta = \quad = 0.9$.

(c) $A = [-15, 10] \times [-11, 11]$, $K = 20$, Jungck-CR iteration with $\alpha = \beta = \quad = 0.5$.

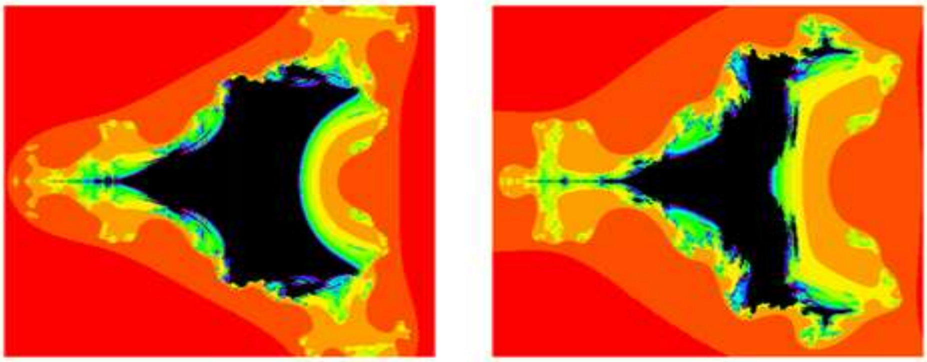
(d) $A = [-100, 60] \times [-61, 61]$, $K = 20$, Jungck-CR iteration with $\alpha = \beta = \quad = 0.1$.

From the obtained images in Figure 5, we can observe that the overall shapes of the sets obtained in the Jungck-CR orbit are completely different than the shape obtained in the Jungck orbit. The shape changes in a significant way with the change of the a, ß and parameters of the Jungck-CR iteration. Moreover, the set in the Jungck orbit has a 5-fold symmetry, whereas the sets in the Jungck-CR orbit lose this type of symmetry. They only exhibit a mirror symmetry with the real axis as the line of symmetry. When we look at the area occupied by the sets, we observe that also in this case depending on the values of the parameters used in the Jungck-CR iteration the sets occupy different areas – for some values they occupy a smaller area and for others a larger.

NEW ESCAPE CRITERIA FOR COMPLEX FRACTALS 1299

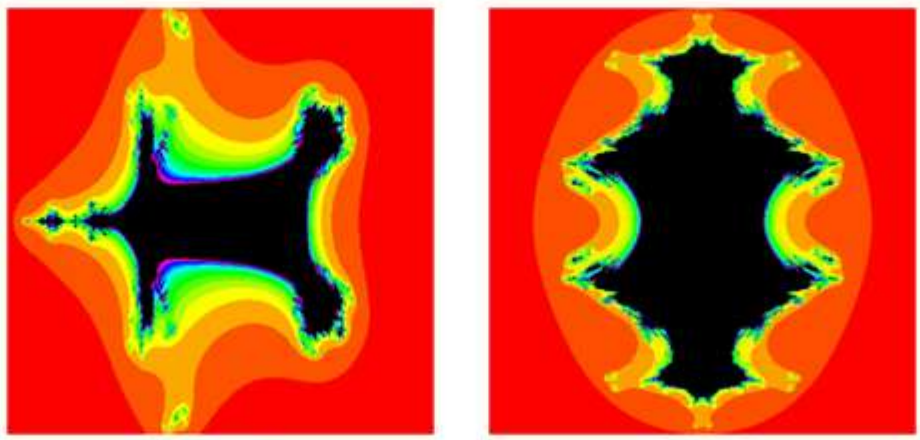


(a) (b)



(c) (d)

Figure 5: Multicorn set for



(a) (b)

Figure 6: Multicorn sets for various parameters in the Jungck-CR orbit

In the second example, we present various multicorn sets generated using the Jungck-CR iteration. In order to generate the images of the sets from Figure 6 the following parameters were used:

(a) $Q_c^*(z) = z^2 + 2z + c$, $A = [-16.5, 8.5] \times [-8, 8]$, $K = 20$, Jungck-CR iteration with $\alpha = 0.6$, $\beta = 0.7$, $= 0.5$.

(b) $Q_c^*(z) = z^3 + (3/2)z + c$, $A = [-2.5, 2.5] \times [-3.5, 3.5]$, $K = 20$, Jungck-CR iteration with $\alpha = 0.9$, $\beta = 0.7$, $= 0.7$.

When we look at the images, we can observe that the shapes of the obtained sets are less inter-esting in comparison to the Mandelbrot and Julia sets. Moreover, we see that both sets have axial symmetries. The image in Figure 6(a) has one axis of symmetry, namely the real axis, whereas the image in Figure 6(b) has two axes of symmetry – the real and imaginary axis.

V. CONCLUSI ONS

In this paper, the Jungck-CR iteration has been introduced in the study of complex fractals (Julia sets, Mandelbrot sets, multicorns). The new escape criterion for the Jungck-CR iteration has been established for the m -th degree complex polynomial functions.

The Jungck-CR does not reduce to the Picard, Jungck-Mann, Jungck-Ishikawa, Jungck-Noor, nor any other iteration studied in the literature on the generation of complex fractals. Thus, the results of this paper open up a new class of complex fractals. Moreover, the obtained complex fractals could further extend the capabilities of the algorithms that use Mandelbrot and Julia sets, e.g. they can expand the domain dictionary used in fractal image compression [42] or broaden the space for the initial keys used in image encryption [41].

In future work, we will attempt to introduce the Jungck-CR and other implicit iteration processes into other types of complex fractals, e.g. fractals obtained through polynomiography.

REFERENCES

[1]. R. Agarwal and V. Agarwal, Dynamic noise perturbed generalized superior Mandelbrot sets, Nonlinear Dynamics , 67 (3) (2012), 1883-1891.

[2]. Ashish, M. Rani, and R. Chugn, Julia sets and Mandelbrot sets in Noor orbit, Applied Mathematics and Computation , 228 (2014), 615-631.

[3]. C. Beck, Physical meaning for Mandelbrot and Julia sets, Physica D, 125 (3-4) (1999), 171-182.

[4]. P. J. Bentley, Methods for improving simulations of biological systems: Systemic computation and fractal proteins,Journal of the Royal Society Interface ,6 (2009), 451-466.

[5]. Y. S. Chauhan, R. Rana, and A. Negi, New Julia sets of Ishikawa iterates, International Journal of Computer Applications, 7(13) (2010), 34-42.

NEW ESCAPE CRITERIA FOR COMPLEX FRACTALS 1301
[6]. W. D. Crowe, R. Hasson, P. J. Rippon, and P. E. D. Strain-Clark, On the structure of the Mandelbar set, Nonlinearity, 2 (4) (1989), 541-553.
[7]. Y. Dang, L. H. Kauffman, and D. Sandin, Hypercomplex iterations: Distance estimation and higher dimensional fractals , World Scientific, Singapore, 2002.
[8]. S. V. Dhurandhar, V. C. Bhavsar, and U. G. Gujar, Analysis of z-plane fractal images from z z+c a for a < 0. Computers & Graphics , 17 (1) (1993), 89-94.
[9]. P. Dom´inguez and N. Fagella, Residual Julia sets of rational and transcendental functions, In: P. J. Rippon and G. M. Stallard, editors, Transcendental dynamics and complex analysis , pages 138-164. Cambridge University Press, 2010.
[10]. P. Fatou, Sur les ´ equations fonctionnelles, Bulletin de la Soci´ et´ e Math ´ematique de France, 47(1919), 161-271.
[11]. M. Garousi and K. Taleslami, Realistic image making with Mandelbrot set, In: C. S. Kaplan and R. Sarhangi, editors, Proceedings of bridges 2009: Mathematics, music, art, architecture, culture , pages 313-314, London, 2009. Tarquin Publications.
[12]. K. Gdawiec, Fractal patterns from the dynamics of combined polynomial root finding methods, Nonlinear Dynamics , 90 (4) (2017), 2457-2479.
[13]. K. Gdawiec and W. Kotarski, Polynomiography for the polynomial infinity norm via Kalantari's formula and nonstandard iterations, Applied Mathematics and Computation , 307 (2017), 17-30.
[14]. K. Gdawiec, W. Kotarski, and A. Lisowska, Biomorphs via modified iterations, Journal of NonlinearScience and Applications, 9(5) (2016), 2305-2315.
[15]. C. J. Griffin and G. C. Joshi, Octonionic Julia sets,Chaos, Solitons & Fractals ,2(1) (1992), 11-24.
[16]. N. Hussain, C. Kumar, and M. A. Kutbi, On rate of convergence of Jungck-type iterative schemes,Abstract and Applied Analysis, 2013: Article ID 132626, 2013.
[17]. G. Julia, M´ emoire sur l'it ´ eration des fonctions rationnelles, Journal de Math´ ematiques Pures et Appliqu´ees, 8(1) (1918), 47-246.
[18]. G. Jungck, Commuting mappings and fixed points,The American Mathematical Monthly, 83(4) (1976),261-263.
[19]. B. Kalantari,Polynomial root-finding and polynomiography , World Scientific, Singapore, 2009.
[20]. S. M. Kang, H. H. Alsulami, A. Rafiq, and A. A. Shahid, S-iteration scheme and polynomiography, Journal of Nonlinear Science and Applications, 8(5) (2015), 617-627.
[21]. S. M. Kang, W. Nazeer, M. Tanveer, and A. A. Shahid, New fixed point results for fractal generation in Jungck Noor orbit with s-convexity, Journal of Function Spaces , 2015: Article ID 963016, 2015.